WHAT WAS A MATHEMATICAL PROBLEM IN ANCIENT CHINA?

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First draft, please do not quote

The oldest extant mathematical sources from China are for the most part composed of particular problems and general algorithms to solve them. The terms of these problems often echoed with the concrete problems that bureaucrats or merchants could confront in their daily practice. It has hence often been assumed that mathematics in ancient China was merely practical, oriented, as it seemed, towards solving concrete problems. I have argued against such a view from many different perspectives. This paper suggests yet another angle from which to reconsider this hasty conclusion.

Its main point consists in questioning whether we really know what a mathematical problem in ancient China was. Historians have often worked under the assumption that the main pieces of scientific text —figure, problem, algorithm— were essentially a-historical objects, which could hence be approached as their present-day counterparts. Interestingly enough, ancient Chinese sources give us evidence to prove that such is not the case.

I shall hence reconsider in a critical way what a problem was in ancient China and gather evidence showing that problems were submitted to a mathematical practice differing, on many points, from the one we spontaneously attach to them. I hope this result will inspire other research works attempting to take a critical distance from all kinds of ancient sources and to find out evidence enabling us to restore various past practices of mathematical problems.

To put my topic in more down-to-earth terms, this paper aims at solving a very simple puzzle. The earliest mathematical writing to have been handed down by the written tradition, the Han Canon *The nine chapters on mathematical procedures*, describes a procedure for multiplying fractions within the context of the following problem:

(**I.19**) "Suppose one has a field which is 4/7 *bu* wide and 3/5 *bu* long. One asks how much the field makes"[1].

Why is it that, in the middle of his commentary on this procedure, the commentator Liu Hui, who completed his commentary in 263, introduced <u>another</u> problem that can be formulated as follows: 1 horse is worth 3/5 *jin* gold. If a person sells 4/7 horse, how much does the person get?

[1] The shape of the field is designated by the name of its dimensions: a field with only a length and a width is rectangular. This passage can be found in [Qian, 1963, vol. 1, p. 99]. Unless otherwise stated, I shall refer the reader to Qian Baocong's edition of *The nine chapters* —[Qian 1963, vol. 1]. Together with Prof. Guo Shuchun (Institute for the history of natural sciences, Academy of science, China), since 1984, I have been working on a critical edition and a French translation of *The nine chapters* and the commentaries. My ideas on these sources were certainly influenced by our collaboration. Problem I.19 is the 19th problem of chapter I. The same convention holds for designating other problems in this paper.

This is the puzzle, for which I shall suggest a solution in this paper. Seen from our point of view, the two problems are identical. Obviously, from Liu Hui's perspective, they differ. Inquiring into their difference as perceived by Liu Hui should hence disclose in which respects the practice of problems in ancient China differs from the one we would be spontaneously tempted to assume. My claim is that it is only when we are in a position to interpret the difference between the two problems that we may think to have elaborated a non-naive reading of ancient China's mathematical problems.

My argument consists in several steps.

First, I shall show that we must discard the obvious explanation that one could be tempted to put forward, namely: that a problem in ancient China only stood for itself. One can find evidence showing that a particular problem was read by the commentators as standing for a class (*lei*) of problems. Moreover, one can prove that these readers expected that an algorithm given by the Canon after a particular problem should allow solving as large a class of problems as possible. In making this point, we shall hence establish the first elements of a description of the practice of problems in ancient China.

This first part thus proves that our puzzle cannot be easily solved and requires another explanation. In a second part, I introduce some basic information concerning the practice of proving the correctness of algorithms as carried out by commentators like Liu Hui, since this will appear to be necessary for our argumentation.

On this basis, the third part argues that, in fact, far from being reduced to statements requiring to be solved, problems are an essential component in the practice of proof. Viewing them from this angle allows accounting for why a problem was substituted for another one.

Before I set out to develop my argumentation, some points require clarification. As is clear from what I just said, the evidence allowing us to describe the practice of problems in ancient China is to be found mainly in commentaries on Canons, the earliest extant ones of which were completed in the 3rd century. In this paper, my argumentation relies especially on Liu Hui's commentary on *The nine chapters on mathematical procedures*, which was composed probably some two or three centuries after the completion of the Canon. The reason why we have to proceed in that way relates to the fact that commentators wrote in a style radically different from that of the Canons. More precisely, commentaries express expectations, motivations and second-order remarks, all these elements being absent from *The nine chapters* —this is how, from now on, I abbreviate the title of the Canon. Commentaries hence allow us to grasp features of mathematical practice that are difficult to approach on the basis of the Canons, at least when one demands that a reading of ancient sources be based on arguments. Naturally, this approach raises a key question: how far can we be sure that what can be established on the basis of 3rd century sources holds true with respect to a Canon composed centuries earlier? Unless we find new sources, there is no way to reach full certainty with respect to this question. However, once we restore the practice of problems to which the commentaries bear witness, we can find many hints indicating that what can be established regarding the practice of problems on the basis of the commentaries holds true for the Canons. Moreover, relying on commentaries composed three centuries after the Canon appears to be less inadequate a method than relying on one's personal experience of a mathematical problem. It seems to me to be more plausible that the practice of problems contemporary with the compilation of *The nine chapters* be related to Liu Hui's practice than it be related to mine.

With these warnings in mind, let us turn to examining our evidence.

1. How does a problem stand for a class of problems?

The example of problem I.19 quoted above illustrates what, in general, a problem included in *The nine chapters* looks like. It is particular in two respects. Its terms refer to a particular, concrete situation, such as, in this case, computing the area of a field. Moreover, the terms mention, for each of the data, a particular numerical value. Again, in our example, the width is 4/7 *bu*, whereas the length is 3/5 *bu*. Some problems are only particular in this latter respect. Moreover, the former criterion should be manipulated with care[2]. However, in general, the terms of most problems of *The nine chapters* conform to this description.

In the case of problem I.19, just mentioned, the procedure that the Canon provides after it is expressed in general and abstract terms, since it reads as follows:

"Procedure for multiplying parts: the denominators being multiplied by one another make the divisor. The numerators being multiplied by one another make the dividend. Divide the dividend by the divisor",[3]

which amounts to:

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

However, in other cases, the procedure given by *The nine chapters* is expressed with respect to the concrete situation and values described by the problem. Such is the case for the following problem, included in chapter IX, "Basis and height (*gougu*)", devoted to the right-angled triangle:

(IX. 9) "Suppose one has a log with a circular section, stuck into a wall and the dimensions of which are unknown. If one saws it with a saw at a depth of 1 *cun* (CD), the path of the saw (AB) is 1 *chi* long. One asks how much the diameter is worth.
Answer: the diameter of the log is worth 2 *chi* 6 *cun*."[4]

The situation of the log stuck into the wall can be represented as follows —note that such a geometrical representation is mine and is not to be found in the ancient sources:

[2] For a general discussion on the form of these problems, see [Chemla 1997]
[3] [Qian, 1963, vol. 1, p. 100].
[4] [Qian 1963, vol. 1, p. 245-6]. I inserted references to the geometrical figure in the translation of the sake of my argument. Needless to say, they are not part of *The nine chapters*.

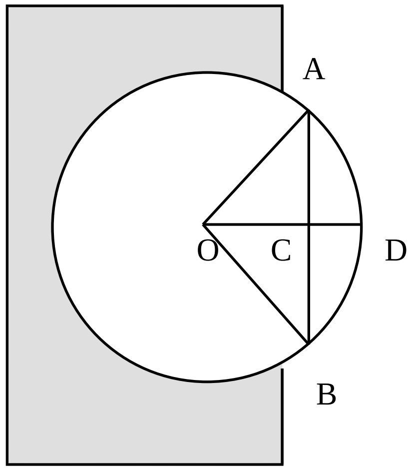

Figure 1

The terms of the problem hence refer to a concrete situation and particular values. The procedure following it, which enables to solve it, is also expressed with reference to them:

“Procedure: half the path of the saw being multiplied by itself, one divides by the depth of 1 *cun*, and increases this by the depth of 1 *cun*, which gives the diameter of the log.”

What is interesting is that, in this latter case, we can find evidence allowing us to capture how Liu Hui read this problem. With this issue in mind, let us examine a passage in which Liu Hui refers to the problem of the log stuck in a wall. This is the commentary Liu Hui inserts after the problem in the framework of which *The nine chapters* deals with the area of the circular segment. If we omit the references to my figure below[5], which I added within brackets, the problem reads as follows:

(I.36) "Suppose again that one has a field in the form of a circular segment, whose chord (AB) is worth 78 *bu* 1/2 *bu*, and whose arrow (CD) is worth 13 *bu* 7/9 *bu*. One asks how much the field makes."[6]

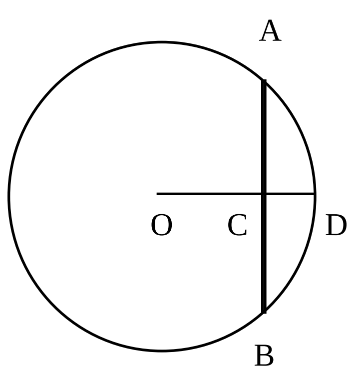

Figure 2

[5] Again, I drew a figure for the sake of my own comments. No such figure is to be found in the original sources.
[6] [Qian, 1963, vol. 1, p. 109-10].

The nine chapters provides an algorithm to compute the area of this field, but Liu Hui criticizes it and sets out to establish a new procedure. It is within this context that Liu Hui needs to rely on the two data of problem I.36, the chord (AB) and the arrow (CD), to compute the diameter of the circle from which the circular segment is cut. For this, he refers to problem IX.9 as follows

"(…) It is advisable then to rely on the procedure of the (problem) where one saws a log with a circular section in the (chapter) "Basis (*gou*) and height (*gu*)" and to look for the diameter of the corresponding (circle) by taking the chord of the circular segment as the length of the path of the saw, and the arrow as the depth of the piece sawn. Once one knows the diameter of the circle, then one can cut the circular segments in pieces (...)"

It causes no surprise to discover, on the basis of this piece of evidence, that Liu Hui does not read the problem of the log stuck in a wall and the procedure attached to it as merely standing for themselves, but as expressing something more general. This reveals that, even in a case as problem IX.9, where the procedure is expressed with reference to the concrete situation of a log stuck in a wall, the 3rd century commentator reads its meaning as exceeding this particular case.

What is more noteworthy, however, is *how* Liu Hui does so. He does not feel the need to express a more abstract statement or procedure that would capture the “essence” of problem IX.9 and could be applied to similar cases such as problem I.36. On the contrary, despite the fact that the concrete situations of I.36 and IX.9 as well as the numerical values are completely different, Liu Hui directly makes use of the procedure given after IX.9, with its own terms, in the context of I.36, by establishing a term-to-term correspondence, “taking the chord of the circular segment as the length of the path of the saw, and the arrow as the depth of the piece sawn.” This seems to indicate that the situation described in problem IX.9 can be directly put into play in other concretely different situations. The particular appears to possibly express the general.

In fact, one can find a similar piece of evidence four centuries later, in a 7th century commentary on the *Mathematical Canon continuing the Ancients* (*Jigu suanjing*), written by Wang Xiaotong in the first half of the 7th century[7]. The commentary discusses the first problem of the book, devoted to astronomical matters, in relation to a problem referred to as included in *The nine chapters* and dealing with a dog pursuing a rabbit. In this case too, the latter problem is not reformulated in either more abstract or astronomical terms for the discussion to develop. Hence, despite the fact that *The nine chapters* again presents the problem within a particular concrete context, the first reader that we can observe, namely the 7th century commentator, reads it as exemplifying a set of problems sharing a similar structure and solved by the same algorithm. Furthermore, like in the previous example, the commentator feels free to make the problem and procedure, which apparently do not relate to astronomy, "circulate" into another different, astronomical, context, without reformulating it either in other concrete terms or in abstract ones. This seems to indicate that an ongoing tradition in ancient China did not mind discussing mathematical procedures in the concrete, particular terms of the problems after which they had been formulated, although the questions discussed exceeded the scope of the particular situation. The evidence displayed dates from the 3rd and the 7th century, and it allows drawing the same conclusions. This indicates that it would not be farfetched to assume that such was also the way in which the authors of *The nine chapters* conceived of the problems with which they were composing the Canon.

[7] [Qian 1963, vol. 2] contains a critical edition of Wang Xiaotong’s *Jigu suanjing*. The problem and commentary mentioned are to be found on [Qian 1963, vol. 2, p. 495-6]. [Eberhard 1997] discusses this example.

In fact, the same conclusions hold true with respect to the numerical values. Although Liu Hui regularly comments on a given problem and procedure on the basis of particular numerical values, one can gather evidence showing that he understands the meaning of his discussion as extending beyond this particular set. Again, in this respect, the commentator thus proves to discuss the general in terms of a particular[8].

The problems of logs stuck in walls and dogs pursuing rabbits that can be found in *The nine chapters* may be perceived as recreational by some readers of today, because of the terms in which they are cast. The evidence examined proves that things are not so simple. The historian is thus warned against the assumption that the category of "mathematical problem" remained invariant in time, and is instead invited to describe the practice of problems with respect to which a text was written, before setting out to read it. Such a historical reconstruction guards us from mistaking a problem as merely particular or practical, when Chinese scholars read it as general and meaningful beyond its own context, or mistaking it as merely recreational when it was put to use in concrete situations. This is a crucial point, since it prevents us from jumping to the conclusion that mathematics in China was merely practical, simply because ancient Chinese texts attest to ways of managing the relationships between abstraction and generality, between pure and practical mathematics, which are different from those we expect.

On the basis of other sections of Liu Hui's commentary, it can be shown that, in his view, a problem stands for a class (*lei*) of problems that is determined on the basis of the procedure described after it. It should be stressed that it is not so much the similarity of structure between the situations described by different problems that allow considering them as sharing the same category, but, most importantly, the fact that they are solved by the same procedure[9]. Far from being the list of operations allowing solving a given problem, the procedure hence appears to be read beyond the specific context within which it is formulated, and it even appears to be that which determines the scope of generality of a given problem.

Now that we have seen how, in Liu Hui's practice, a problem did practically stand for a category of problems, we have discarded the simple solution that could have accounted for our puzzle. We are hence left with the question: Why is it that, within the context of his commentary on the "procedure for multiplying parts", Liu Hui feels it necessary to substitute a problem for another one, although both seem to us to share the same category? Elaborating an interpretation for this fact will compel us to enter more deeply into the practice of mathematical problems as Liu Hui's commentary bears witness to it.

2. Proving the correctness of procedures: the operations of homogenization and equalization.

Interpreting Liu Hui's commentary on the "procedure for multiplying parts" requires recalling some basic information regarding the mathematical practices that are linked to the exegesis of such a Canon as *The nine chapters*.

In fact, Liu Hui inserts a commentary after virtually any procedure given by *The nine chapters* for solving a problem or a set of problems. And, in each of these, the commentator

[8] See my "Generality rather than abstraction" (in preparation).

[9] On this point, I refer the reader to [Chemla 1997]. As rightly stressed by C. Cullen, the concept and practice of "categories" in mathematics is central to the opening sections of the *Zhoubi suanjing* [Cullen, 1996, pp. 74-5 & 177]. In the glossary of mathematical terms that I composed to be published together with [Chemla, Guo forthcoming], I discuss the various uses of the term *lei* in the commentaries on *The nine chapters.*

systematically establishes the correctness of the procedure described by the Canon. However, the way in which he tackles this question manifests a specific practice of proof, which can be linked to the context of exegesis within which it develops[10]. I shall sketch its main characteristics, since it will prove useful for solving our puzzle.

To this end, I shall evoke Liu Hui's commentary on the algorithm given by *The nine chapters* to add up fractions, which appears to be a pivotal section in his text[11].

In chapter I, where the arithmetic of fractions is fully treated, the Canon sets three problems requiring to add up fractions, after which it offers the following general algorithm:

"Procedure for gathering parts:

The denominators multiply the numerators that do not correspond to them; add up; take this as the dividend (*shi*).

The denominators being multiplied by one another make the divisor (*fa*).

Divide the dividend by the divisor.

If the denominators are equal, one adds them (the numerators) directly with each other"[12], i.e.:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$$

or a direct addition, in case $b = b'$.

Let us outline how, in his commentary on this section of the Canon, Liu Hui establishes the correctness of this procedure.

To this end, we first need to outline how both the Canon and the commentaries approached the mathematical objects involved, namely: the fractions. The expression for *m/n* used by *The nine chapters*, "m of n parts" (n *fen zhi* m), gives the fraction as being composed of "parts". It also displays a numerator and a denominator. Both approaches of fractions appear to have been combined by the mathematicians of ancient China. On one hand, the problem asking to add up fractions gathers various disparate parts together to form a quantity, which must hence be evaluated. On the other hand, the algorithm prescribes computations on numerators and denominators to get a fraction. Establishing the correctness requires proving that the fraction obtained measures the quantity formed.

In a previous section, approaching the fractions as manipulated by the algorithm, i.e.: as a pair of numerator and denominator, Liu Hui had stressed the potential variability of their expression: one can multiply, or divide, the numerator and the denominator by a same number, he stated, without changing the quantity meant. In this context, to divide is to simplify the fraction. The opposed operation, to "complicate", which Liu Hui introduced, is needed only for the sake of the proofs.

[10] I address the question of the relationship between the exegesis of a Canon and Liu Hui's specific practice of proof in [Chemla forthcoming]

[11] I discuss in detail this section of Liu Hui's commentary in [Chemla 1997-a].

[12] For this section of the Canon and Liu Hui's commentary, see [Qian, 1963, vol. 1, p. 95-6].

At the beginning of his commentary on the "procedure for gathering parts, Liu Hui, then, considers the counterpart of these operations with respect to the fractions regarded as parts: simplified fractions correspond to coarser parts, complicated ones to finer ones. At this level, Liu Hui again stresses the invariability of the quantity, beyond possible changes in the way of composing it.

Now to prove that

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$$

the commentator shows that the algorithm amounts to, by multiplication of both the numerator and the denominator, refining the disparate parts so as to make them share the same size. Quoting the Canon, he expounds the actual meaning of each step, in terms of, both, parts and numerators/denominators, making clear how they combine to fulfil the program outlined. When "the denominators are multiplied by one another", an operation that, in the course of the proof, he names "to equalize", this computes the denominator common to all fractions and defines the size that the different parts can share: they can thus be added. Moreover, when "the denominators multiply the numerators that do not correspond to them", to yield *ad* and *bc*, the numerators, he says, are made homogeneous with the denominators to which they correspond, and hence the original quantities are not lost. Here too, in passing, he confers a name to this set of operations: "to homogenize".

"Equalizing" the denominators and "homogenizing" the numerators, the algorithm thus actually yields a correct measure of the quantity formed by joining the various fractions.

We, contemporary readers, read this text as establishing the correctness of an algorithm. But what are the aims pursued by Liu Hui when writing it? In fact, in this case, they are highlighted by the following part of this commentary, since, at this point, far from ending, his comment goes on with highly abstract and philosophical considerations, concluded by a key declaration: "Multiply to disaggregate them, simplify to assemble them, homogenize and equalize to make them communicate, how could those not be the key-points of computations/mathematics (*suan*)?"

Opaque as it may seem, this declaration is essential. It clearly shows that something else is at stake in the previous proof than establishing the correctness of a procedure. The proof had exhibited operations at play in the algorithm (multiplying, dividing, equalizing, homogenizing), which precisely constitute the topic of the subsequent considerations: exhibiting them appears to be one concern at stake in carrying out the proof. Moreover, "equalizing", "homogenizing" were introduced in relation to fractions, where they referred to precise operations on numerators and denominators. However, Liu Hui's concluding declaration indicates that their relevance far exceeds this limited context, since they are now listed among the "key-points of mathematics". How are we to interpret these facts?

In fact, when reading the whole commentary, we see these operations recurring in several other proofs offered by Liu Hui for establishing the correctness of other procedures described in the Canon. This fact explains why Liu Hui's declaration can be so general. However, understanding how the operations of "homogenizing" and "equalizing" can have a general validity requires interpreting them in a new way.

In the context of adding up fractions, equalizing was interpreted as the operation equalizing denominators whereas homogenizing was interpreted as that which computed numerators homogeneous with the new denominators. In the other contexts in which they can be met, the operations of equalizing and homogenizing can also be interpreted in terms of their local effects. However, the fact that they recur in different contexts reveals a second, formal, general meaning for both terms, relating to *how* the algorithms work. All algorithms in which these operations are shown to be at play proceed by equalizing some quantities while homogenizing others. The two terms capture and express the strategy followed by the procedure. And the parallel between the proofs discloses that, in fact, the algorithms follow the same formal strategy of equalizing and homogenizing. Even though the concrete meanings of equalizing and homogenizing vary according to the contexts in which they are at play, formally they operate in the same way. The proofs bring to light that the same fundamental algorithm underlies various procedures. It is on the basis of the actual reasons accounting for the correctness of the algorithms, that, through the proofs, a concealed formal connection between them is unveiled.

This relates to one of the reasons for Liu Hui to carry out proofs, i.e.: bringing to light such fundamental formal strategies common to the various procedures provided by *The nine chapters.* Such key algorithms allow reducing the variety of procedures of the Canon, uncovering a small number of strategies systematically put to use in designing all its procedures. Multiplying or dividing all numbers in an adequate set, equalizing and homogenizing appear to underlie many of the algorithms of the Canon: this is why, when they first occur, in the context of the "procedure for gathering parts", Liu Hui immediately stresses their importance. His declaration appears to gather the most fundamental algorithms underlying the procedures of *The nine chapters,* as brought to light by the proofs carried out in his commentary.

These operations are also at play in the "procedure for multiplying parts", and we shall see that the problems play a key part in the way in which Liu Hui uncovers them in this case. We are thus now in a position to go back to our puzzle and offer a solution for it.

3. The problems as a condition for exhibiting formal strategies.

As already mentioned, the problem in the context of which the "procedure for multiplying parts" is given requires computing the area of a rectangular field, with length 3/5 and width 4/7. The procedure itself is expressed with a great generality, since, as we saw, it reads:

"Procedure for multiplying parts: the denominators being multiplied by one another make the divisor. The numerators being multiplied by one another make the dividend. Divide the dividend by the divisor".

In a first part of his commentary on it, Liu Hui develops an abstract reasoning accounting for the correctness of the procedure. This argumentation shows how multiplication and division are at play in the design of the procedure. It can be translated as follows:

"In each of the case when a dividend does not fill up a divisor[13], they hence have the name of numerator and denominator[14]. If there are parts, by expanding the corresponding dividend by

[13] This technical expression refers to the case when the dividend is smaller than the divisor. Beware that "dividend" designates the content of a position on the counting board, and not a determined number —this is the assignment of variables, characteristic of the description of algorithms in ancient China. In what follows, the same word will hence designate different values, depending on the operations that have been applied to it.

[14] The result of the division is hence the fraction, whose numerator and denominator are respectively the dividend and the divisor. "Numerator" and "denominator" refer to the numbers as constituting a fraction; "dividend" and "divisor" refer to them as the terms of the operation yielding the fraction. The commentary alternately uses the two sets of terms, with the greatest precision.

multiplication, then it may happen that it (the dividend produced by the multiplication) fills up the divisor and that the (division) hence only yields an integer[15]. If, furthermore, one multiplies something by the numerator, the denominator must consequently divide (the product) in return (*baochu*). "Dividing in return", this is dividing the dividend by the divisor".

Before reading the end of the argument, let us make some comments at this point. The expression of "dividing in return" (*baochu*) is particularly important to stress. This occurrence constitutes its first one in the book. As an operation, as the commentator defines it, it consists in a division. However, the qualification "in return" makes explicit the <u>reason</u> for dividing. Using "dividing in return" implies that, for some reason, one has beforehand used a superfluous multiplication, and that, by this division, one deletes its effect. This can be clearly understood in our case. Within the context of the previous sentences, multiplying by the <u>numerator</u> is to be interpreted as follows: instead of multiplying "something" by a fraction a/b, one multiplied the "something" by its numerator a. It is in multiplying the fraction by b that one yields the numerator a. Having multiplied the "something" by a, the numerator, instead of a/b, one has multiplied by a value that was b times bigger than what was desired. Consequently, one has to "divide in return" by that with which one had multiplied beforehand, namely: b. Multiplying something by a/b is hence shown to be the same as multiplying by a alone, and dividing the result by b.

"Dividing in return" is one of the qualifications of division that one can find in *The nine chapters*. Two points are important about it. First, the expression clearly adheres to the sphere of justification, since the very formulation of the division indicates a reason for carrying it out. Secondly, we can find the expression <u>both</u> in *The nine chapters*, and the commentaries. This point will be useful later in our argumentation. However, we shall soon meet with another similar qualification, which is to be found only in the commentaries.

The following sentence, in this section of Liu Hui's commentary, goes on formulating the argument establishing the correctness of the procedure, by quoting one of the operations prescribed by the Canon:

"Now, "the numerators are multiplied by one another"[16], the denominators must hence each divide in return"

For each of the numerators of the fractions to be multiplied, one can apply the argument developed above. The consequence is that if one multiplies the numerators instead of multiplying the fractions, one must hence divide the product by each of the denominators. The last sentence transforms this sequence of operations just established to carry out the multiplication of fractions into the one supplied by the Canon, as follows:

"Consequently, one makes the "denominators multiply by one another" and one divides at a stroke (*lianchu*) (by their product)".

Transforming a sequence of two divisions into a unique division by the product of the divisors is a valid transformation because the results of division are exact. Liu Hui proves to be

[15] If the division of a by b yields a/b, multiplying a by kb, the new dividend akb divided by b yields the integer ak.
[16] This is the quotation from the procedure of the Canon.

attentive to this fact[17]. As above, the formulation "to divide at a stroke" prescribes a division, in a way that indicates the reason why the division can be carried out in this way. This expression recurs regularly in Liu Hui's commentary. The qualification adheres to the sphere of justification. In contrast to the "division in return", however, the expression *lianchu* never occurs in *The nine chapters*.

This conclusion ends the justification of the correctness of the procedure. One can see how multiplication and division, two of the fundamental operations listed in Liu Hui's declaration, are shown to be put into play, as opposed operations, for designing the procedure. For our purpose, it is interesting to note that the proof of the correctness of the algorithm develops here independently of the framework of the problems in the context of which the "procedure for multiplying parts" was formulated. The arguments only made use of general properties of dividend and divisor, numerator and denominator.

However, Liu Hui does not end here his commentary on this procedure. He goes on to highlight how, seen from another angle, this procedure also puts into play equalizing and homogenizing, and this is the very point where we go back to the question raised at the beginning of this paper.

The sentence linking the two parts of the commentary is essential to examine for our purpose. Liu Hui states:

"If here one makes use of the formulation of a field with length and width, it is difficult to have (the procedure) <u>be understood in all its dimensions</u> (*nan yi guang yu*)."[18]

In fact, this sentence justifies discarding the problem in the context of which the "procedure for multiplying parts" is described in *The nine chapters* and introducing, as is done immediately after, another type of problem equivalent to the one mentioned above: "1 horse is worth 3/5 *jin* gold. If a person sells 4/7 horse, how much does the person get?" We shall examine below how other mathematical problems are put into play in the second part of the commentary. For the moment, let us inspect more closely the previous assertion.

Liu Hui's statement establishes a link between the context of a problem, —here the problem provided by *The nine chapters*, i.e., that of a rectangular field—, and the will of <u>understanding</u> the procedure. It appears here that a given problem can impede gaining a <u>fuller</u> understanding of the procedure. This seems to indicate that the preceding passage is also perceived by Liu Hui as contributing to an understanding of the procedure. This link between establishing the correctness of a procedure and understanding it is not fortuitous. It will recur again in the following passage.

In addition to this, the change of problem is justified by the attempt of gaining a better understanding of the procedure. Strikingly enough, this indicates a possible link between the context of a problem and the proof of the correctness of a procedure. Again, this link will be confirmed in what follows. In fact, the new problems introduced enable developing a second proof of the correctness of the procedure that also conforms to the practice of demonstration sketched above. Let us stress at once the essential consequence for us: problems appear not to be reduced to questions that require a solution, but they also play a part in proofs.

Observing Liu Hui's next development should hence allow us progressing on two fronts. It should provide evidence showing how a problem intervenes in helping understand a procedure.

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[17] On this point, see [Chemla 1997/8].

[18] The passage of Liu Hui examined below can be found in [Qian, 1963, vol. 1, p. 100].

And this is where the difference between the two problems should appear. Moreover, it should help us grasp how Liu Hui conceives of the fact of “understanding a procedure”.

Let us first indicate, in modern terms, the main idea of the second proof of the correctness of the procedure. We shall then be in a position to grasp the difference between our two problems. Thereafter, we shall observe in detail how the proof actually makes use of the context of problems.

The main idea of the second proof can be represented in modern terms by the following sequence of expressions:

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bc} \cdot \frac{bc}{bd} = \frac{ac}{bd}$$

It then brings to light that one can yield the algorithm by making use of the possibility of “complicating” fractions in such a way as to equalize the denominator of the first fraction and the numerator of the second one. Both being equal to *bc*, the sequence of operations can then be reduced to dividing *ac* by *bd*, which amounts to the algorithm the correctness of which is to be proved. In fact, this last step requires a justification, which Liu Hui provides by interpreting the results of all operations in the context of the situation described by a problem. And, here, as we shall show below, we introduce the essential factor: Liu Hui’s proofs regularly make use of interpreting the intentions of the operations of an algorithm in terms of the situation described by the problem.

This rough presentation of the second proof enables us to understand why the first problem cannot be used here. The situation of the field with length and width does not offer rich enough a semantic field for the computation of *bc* to be interpreted in a natural way. In other terms, the problem of computing the area of a rectangular field does not allow bringing to light the equalization underlying the “procedure for multiplying parts”. The scheme of “equalizing” and “homogenizing” cannot be unfolded in this context.

In contrast to it, with the three components that are the quantities of gold, person, and horse, the second problem offers richer possibilities for interpreting the effects of operations. And this semantic field is thus rich enough for disclosing that the scheme of “homogenizing” and “equalizing” underlies the “procedure for multiplying parts”. Allowing to fully developing the scheme of “equalizing” and “homogenizing”, the problem enables to show in another way how the “procedure for multiplying parts” relates to the fundamental operations identified by Liu Hui. The key point here is that the interest in bringing to light that equalization is at play belongs only to the sphere of proving. The equalization plays no role in the “procedure for multiplying parts” itself. Its silent operation in the procedure can be disclosed only if the situation described by the terms of the problem is rich enough. This is why the problem in the context of which the algorithm is described differs from the problem in relation to which the proof is carried out. A link is thereby established between bringing to light a formal strategy accounting for the correctness of the procedure and interpreting the intentions of the operations in the semantic field provided by the terms of a problem. This situation, where describing the algorithm and carrying out its proof require different problems, reveals the essential part played by problems in establishing the correctness of algorithms. Note, however, that the values are common to both problems: Liu Hui introduced the second problem in such a way that the question to be solved remains to multiply 3/5 by 4/7. This implies that the computations linked to solving the first problem are an actual subset of the operations involved in proving the correctness of the "procedure for multiplying parts".

With these observations in mind, let us examine how Liu Hui makes use of problems in his second proof. In fact, this part of his commentary consists in articulating, in an adequate way, a sequence of equivalent problems that are transformed one into the other. What would be done in modern mathematics by formal computations is here carried out by interpreting operations with respect to a succession of semantic fields.

In this second proof, Liu Hui's first step consists in formulating a first problem, which is characterized by the fact that its solution requires making use of the last operation of the "procedure for multiplying parts", i.e.: dividing 12 (*ac*) by 35 (*bd*). It reads as follows:

"Suppose that one asks: 20 (*bc*) horses are worth 12 (*ac*) *jin* gold. If one sells the 20 (*bc*) horses and if 35 (*bd*) persons share [the gain], how much does a person get? Answer: 12/35 *jin.* To solve it, one must follow the "procedure for sharing parts[19]" and take 12 *jin* gold as dividend and 35 persons as divisor."

The division prescribed is exactly the last step of the "procedure for multiplying parts", both in terms of values and operations. This problem is immediately followed by another problem, given to be a transformation of the former one, and characterized by the fact that its solution requires using the whole list of operations prescribed by the "procedure for multiplying parts". The key point is that the procedure for solving this second problem is first described in terms of "homogenizing" and "equalizing".

"Suppose that, modifying (the problem), one says: 5 (*b*) horses are worth 3 (*a*) *jin* gold. If one sells 4 (*c*) horses and if 7(*d*) persons share [the gain], how much does a person get? Answer: 12/35 *jin*.

"To solve it, one has to <u>homogenize</u> these quantities of persons (*bd*) and gold *(ac*), and this then conforms with the first problem" and is solved by the "(procedure for) sharing parts"."

The use of the term "homogenize" implies that, in parallel, an equalization is carried out. Only the operation of equalization can confer the meaning of "homogenization" to the other operations. Liu Hui will make this point explicitly later on in the passage. "Homogenizing" quantities of gold and persons goes along with equalizing quantities of horses in both statements of the problem. This points to the production of the new following statements, equivalent to the first problem:

20 (*bc*) horses are worth 12 (*ac*) *jin* gold.

If one sells 20 (*bc*) horses and if 35(*bd*) persons share [the gain], how much does a person get?

Transforming the second problem into the first one requires computing *ac* and *bd*, after what the former ought to be divided by the latter. This sequence corresponds step by step to the "procedure for multiplying parts". This hence allows Liu Hui, in the next sentence, identifying

[19] In fact, this procedure, which is described just before the "procedure for multiplying parts", covers all possible cases of division involving integers and fractions. See [Chemla 1987]. To make my argument clearer, I insert in the translation modern symbolic expressions.

both procedures and thereby transferring the interpretation in terms of homogenization and equalization onto the “procedure for multiplying parts”. He writes:

“If this is so, “multiplying the numerators by one another to make the dividend (*ac*)”, is like homogenizing the corresponding gold.

“Multiplying the denominators by one another to make the divisor (*bd*)” is like homogenizing the corresponding persons.

“Equalizing the corresponding denominators makes 20 (*bc*). But the fact that the horses be equalized plays no role. One only wants to find the homogenized and this is all.”

It is now clear that the “procedure for multiplying parts” involves only homogenizations. As Liu Hui stresses, the equalization plays no part in the procedure itself, except for interpreting its first steps as “homogenizations”. This can only be done within the semantical field of the new problem, and could not been carried out within the framework of computing the area of a rectangular field.

In conclusion, Liu Hui makes the last point enabling him to formulate a problem in terms of horses, gold and persons strictly equivalent to the problem provided by *The nine chapters*. He states:

“Moreover, that 5 horses are worth 3 *jin* gold, these are *lü* in integers. If one expresses them in parts, then this makes that one horse is worth 3/5 *jin* gold. That 7 persons sell 4 horses, that is that 1 person sells 4/7 horse”

Qualifying the data in each statement as *lü* designates the property of the pairs to be possibly multiplied or divided by a same number, without the meaning of the relationship be modified[20]. Drawing the consequence of this statement to transform the terms of the latter problem yields the following problem, which is the one with which we formulated our puzzle:

1 horse is worth 3/5 (*a/b*) *jin* gold. If one sells 4/7 (*c/d*) horse, how much does the person get?

How can we qualify the understanding of the procedure produced by Liu Hui’s development? In fact, as was the case for the “procedure for gathering parts”, Liu Hui proves in a second way the correctness of the procedure by bringing to light its relation to the procedure of homogenizing and equalizing. Let us stress, however, that the meaning of equalizing and homogenizing differ for both procedures. In the procedure for gathering parts, equalizing meant equalizing the denominators. Here, equalizing refers to the fact that a denominator and a numerator are made equal for the procedure to perform its task. Even though, the concrete meaning of equalizing and homogenizing differs according to the context, it is still the same formal strategy that is put into play in both cases. In both contexts, il is by making some quantities equal and other quantities homogeneous with them that one can yield the solution. And this is what the proofs show. This remark indicates in which sense the fundamental operations listed by Liu Hui can be deemed fundamental for mathematics.

Liu Hui’s proof highlights that, formally, the first operations prescribed by the “procedure for multiplying parts” amount to homogenizing some quantities. The proof hence discloses a new meaning for the algorithm, another way of conceiving of its formal strategy. Each of the proofs

[20] On this concept, see [Guo 1984].

sheds a different light on the “procedure for multiplying parts”. The new problems introduced appear to be an essential condition for bringing the pattern of homogenizing and equalizing to light, since they make possible that the equalizing appears. It is in this way that we can understand Liu Hui’s introductory statement, that he needed to discard the problem of the area of the rectangular field to “have (the procedure) be understood in all its dimensions”.

The second proof introduced a procedure of homogenization and equalization in which the “procedure for multiplying parts” is embedded and which provides a new interpretation for it. This procedure of homogenization and equalization required the formulation of a new problem. The difference between this new problem and the one described by *The nine chapters* now precisely appears to lie in the fact that equalization can be interpreted in the new problem, but not in the old one. Problems play a key part in Liu Hui’s practice of proof, and this is what accounts for the fact that problems that seemed the same to us are in fact different for him.

This is how I suggest that our puzzle can be solved. Its solution reveals that, in ancient China’s mathematical practice, problems did not boil down to being statements requiring a solution. They were also used as providing a semantic field that could be put into play to interpret the operations of an algorithm and establish its correctness. This aspect appears to be an essential component of the practice of proof as carried out by the commentators. What is most interesting, furthermore, is what the case discussed here shows: the interpretation of the operations in the semantic field of a problem is used to carry out the proof even in its most formal dimensions.

Two points should be added here.

First, in the same way as, as we showed in our first part, a problem stands for a class of problems sharing the same category, without requiring the development of an abstract formulation, the new problem with horses, gold and persons, each with particular values, also stands for a class of problems. We stated above that it was the procedure solving a problem that was the basis for defining the category (*lei*) of the problem. In this case, all problems similar to the one with horses, gold and persons, in which the procedure of the proof, i.e.: equalization and homogenization, can be unfolded share the same category. The proof can be discussed in terms of horse, gold and person, with particular values, but is most probably meant to be read as general. The general is discussed in terms of the particular here as above, when the problem of the piece sawn was used within the context of the field with the shape of a circular segment. Moreover, the same conclusion can be drawn not only for the situation of a problem, but also for the particular values it involves. Proofs relying on problems with particular values are also meant to be general in this sense: the case of the pyramid with square basis (*yangma*) is a clear illustration for it.

Secondly, it is now time to go back to the problems as a component of *The nine chapters*. All that was said so far mainly relied on the evidence provided by the commentaries. Can one determine whether the compilers of the Canon also had such a use of problems? Let us repeat that, unless new sources are found, we will not be able to answer this question with full certainty. However, some hints can be given.

It can first be noticed that, contrary to what is often said, the Canon manifests an interest in the reason why the procedures are correct. Let us only indicate one fact that supports this hypothesis. We noticed above that the qualification of division as “dividing in return (*baochu*)” adhered to the sphere of justifying procedures. It is hence quite interesting that this expression occurs several times in *The nine chapters*. Why should one prescribe to “divide in return” instead of “divide”, were it not to indicate the reason for carrying out the operation?

My second hint relies again on Liu Hui’s commentary, but in another way. I claim that Liu Hui himself believed that, if the editors of *The nine chapters* described procedures in the context of problems, that related to the use of problems we discussed in this paper. Here is the argument with which I support this claim.

In fact, there are cases in the Canon where procedures are described without the context of problems. Such is the case, for instance, for the rule of three (*jinyoushu*). It is interesting, incidentally, that, in order to establish its correctness, Liu Hui's commentary introduces a problem allowing interpreting the intentions of the operations. What is most interesting in this case lies elsewhere. The commentator states about this procedure: "This is a general procedure (*ci doushu ye*)"[21]. There is only one other passage in which Liu Hui makes a similar comment. It applies to the procedure for solving systems of simultaneous linear equations (*fangchengshu*). However, this "general procedure" is, in this case, described by *The nine chapters* within the context of a problem of millets yielding grains. In fact, immediately after having qualified the procedure as "general", Liu Hui adds a statement that can be interpreted as accounting for why, here, in contrast to the case of the rule of three, the Canon makes use of the context of problems to present the procedure. He writes: "It would be difficult to understand (the procedure) with abstract expressions (*kongyan*), this is why one deliberately links it to (the case of) millets to eliminate the obstacle"[22]. This indicates that, in Liu Hui's perspective, the purpose of the Canon for presenting a procedure in the context of a problem related to the will of having the procedure be understood. We showed the relationship between "understanding" a procedure and establishing its correctness. This piece of evidence may hence indicate that Liu Hui reads the Canon as providing problems to be put into play in proving algorithms.

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